THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Classwork 2

May 24, 2017

Name:

1. (40 points) Let $(c, 0)$ and $(-c, 0)$ be given and let H be the set of all points (x, y) whose difference in distances to $(c, 0)$ and $(-c, 0)$ is a constant $2a$.

   (a) Show that H is the hyperbola
   $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ .$$

   (b) Show that it admits the parametric equations
   $$x = \pm a \cosh t, \quad y = b \sinh t, \quad t \in \mathbb{R} \ .$$

   Solution. (a) From
   $$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a \ ,$$
   we get $x^2/a^2 - y^2/b^2 = 1$ as in the case of ellipse. (b) is a direct check.

2. (30 points) The folium of Descartes in parametric form is given by
   $$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}, \quad a > 0 \ .$$

   (a) Show that it defines a regular curve on $(-\infty, -1)$ and $(-1, \infty)$.

   (b) Verify that it is the solution set to
   $$x^3 + y^3 = 3axy \ .$$

   Solution.

   (a) We differentiate x, y to get
   $$x'(t) = \frac{3a - 6at^3}{(1+t^3)^2} \ , \quad y'(t) = \frac{6at - 3at^3}{(1+t^3)^2} \ .$$
   One finds that $(x'(t), y'(t)) \neq (0, 0)$ for all $t \neq 1$. It defines a regular curve.

   (b) When $x = \dfrac{3at}{1+t^3}, y = \dfrac{3at^2}{1+t^3}$, we have
   $$\begin{aligned} x^3 + y^3 &= \frac{27a^3t^3(1+t^3)}{(1+t^3)^3} \\ &= \frac{27a^3t^3}{(1+t^3)^2} \\ &= 3axy \ . \end{aligned}$$

3. (30 points) Find the position $\mathbf{x}(t)$ of the motion in space when the acceleration and initial data are specified by:

$$\mathbf{a}(t) = (\cos t, \sin t, 1); \quad \mathbf{x}(0) = (100, 20, 0), \quad \mathbf{v}(0) = (0, 0, 5) \ .$$

Solution.

$\mathbf{a}(t) = (\cos t, \sin t, 1)$. Integrating $\mathbf{a}(t)$ to get $\mathbf{v}(t)$:

$$\begin{aligned}\mathbf{v}(t) &= \int_0^t \mathbf{a}(\tau) d\tau + \mathbf{v}(0) \\ &= (\sin t, -\cos t + 1, t - 0) + (0, 0, 5) \\ &= (\sin t, -\cos t + 1, t + 5).\end{aligned}$$

Again, Integrating $\mathbf{v}(t)$ to get $\mathbf{x}(t)$:

$$\begin{aligned}\mathbf{x}(t) &= \int_0^t \mathbf{v}(\tau) d\tau + \mathbf{x}(0) \\ &= (-\cos t + 1, -\sin t + t, \frac{t^2}{2} + 5t - 0) + (100, 20, 0) \\ &= (-\cos t + 101, -\sin t + t + 20, \frac{t^2}{2} + 5t).\end{aligned}$$